

Asymptotic description of stochastic neural networks. II - Characterization of the limit law

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Abstract

We continue the development, started in [2], of the asymptotic description of certain stochastic neural networks. We use the Large Deviation Principle (LDP) and the good rate function H announced there to prove that H has a unique minimum μ_e , a stationary measure on the set of trajectories $\mathcal{T}^{\mathbb{Z}}$. We characterize this measure by its two marginals, at time 0, and from time 1 to T . The second marginal is a stationary Gaussian measure. With an eye on applications, we show that its mean and covariance operator can be inductively computed. Finally we use the LDP to establish various convergence results, averaged and quenched.

Résumé

Description asymptotique de réseaux de neurones stochastiques. II - caractérisation de la loi limite

Nous prolongeons le développement, commencé en [2], de la description asymptotique de certains réseaux de neurones stochastiques. Nous utilisons le Principe de Grandes Déviations (PGD) et la bonne fonction de taux H que nous y annonçons pour démontrer l'existence d'un unique minimum, μ_e de H , une mesure stationnaire sur l'ensemble $\mathcal{T}^{\mathbb{Z}}$ des trajectoires. Nous caractérisons cette mesure par ses deux marginales, à l'instant 0, et du temps 1 au temps T . La seconde marginale est une mesure gaussienne stationnaire. Avec un oeil sur les applications, nous montrons comment calculer de manière inductive sa moyenne et son opérateur de covariance. Nous montrons aussi comment utiliser le PGD pour établir des résultats de convergence en moyenne et presque sûrement.

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Version française abrégée

Après avoir rappelé dans la section 2 les notations et le modèle de réseaux de neurones utilisés dans [2], nous montrons dans la proposition 3.1 et le théorème 3.2 que la bonne fonction de taux H du PGD annoncé dans cette publication admet un minimum unique. Le théorème 3.2 fournit une méthode constructive de calcul effectif de la loi de ce minimum. Nous montrons enfin, dans la section 4, l'intérêt de ce minimum qui apparaît comme la limite faible quand $n \rightarrow \infty$ (le nombre de neurones tend vers l'infini) de la loi Q^{V_n} du réseau moyennée par rapport aux poids synaptiques, c'est un résultat en moyenne. Nous montrons aussi dans le corollaire 4.3 un résultat de convergence faible presque sûrement par rapport aux poids synaptiques, résultat intéressant d'un point de vue pratique puisqu'il évite de prendre la moyenne par rapport à tous les réseaux. Le théorème 4.4 donne un résultat de convergence presque sûre de la mesure empirique vers le minimum de H .

1 Introduction

In [2] we started our asymptotic analysis of very large networks of neurons with correlated synaptic weights. We showed that the image Π^n of the averaged law Q^{V_n} through the empirical measure satisfied a large deviation principle with good rate function H . In the same article we provided an analytical expression of this rate function in terms of the spectral representation of certain Gaussian processes. In the next section we recall some definitions given in [2].

2 Mathematical framework

For some topological space Ω equipped with its Borelian σ -algebra $\mathcal{B}(\Omega)$, we denote the set of all probability measures by $\mathcal{M}(\Omega)$. We equip $\mathcal{M}(\Omega)$ with the topology of weak convergence. For some positive integer $n > 0$, we let $V_n = \{j \in \mathbb{Z} : |j| \leq n\}$, and $|V_n| = 2n + 1$. Let $\mathcal{T} = \mathbb{R}^{T+1}$, for some positive integer T . We equip \mathcal{T} with the Euclidean topology, $\mathcal{T}^{\mathbb{Z}}$ with the cylindrical topology, and denote the Borelian σ -algebra generated by this topology by $\mathcal{B}(\mathcal{T}^{\mathbb{Z}})$. For some $\mu \in \mathcal{M}(\mathcal{T}^{\mathbb{Z}})$ governing a process $(X^j)_{j \in \mathbb{Z}}$, we let $\mu^{V_n} \in \mathcal{M}(\mathcal{T}^{V_n})$ denote the marginal governing $(X^j)_{j \in V_n}$. For some $X \in \mathcal{T}$ and $0 \leq a \leq b \leq T$, $X_{a,b}$ denotes the $b - a + 1$ -dimensional subvector of X . We let $\mu_{a,b} \in \mathcal{M}(\mathcal{T}_{a,b}^{\mathbb{Z}})$ denote the marginal governing $(X_{a,b}^j)_{j \in \mathbb{Z}}$. For some $j \in \mathbb{Z}$, let the shift operator $\mathcal{S}^j : \mathcal{T}^{\mathbb{Z}} \rightarrow \mathcal{T}^{\mathbb{Z}}$ be $\mathcal{S}^j(\omega)^k = \omega^{j+k}$. We let $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ be the set of all stationary probability measures μ on $(\mathcal{T}^{\mathbb{Z}}, \mathcal{B}(\mathcal{T}^{\mathbb{Z}}))$ such that for all $j \in \mathbb{Z}$, $\mu \circ (\mathcal{S}^j)^{-1} = \mu$. Let $p_n : \mathcal{T}^{V_n} \rightarrow \mathcal{T}^{\mathbb{Z}}$ be such that $p_n(\omega)^k = \omega^{k \bmod V_n}$. Here, and throughout the

paper, we take $k \bmod V_n$ to be the element $l \in V_n$ such that $l = k \bmod |V_n|$. Define the process-level empirical measure $\hat{\mu}_n : \mathcal{T}^{V_n} \rightarrow \mathcal{M}_S(\mathcal{T}^{\mathbb{Z}})$ as

$$\hat{\mu}_n(\omega) = \frac{1}{|V_n|} \sum_{k \in V_n} \delta_{S^k p_n(\omega)}. \quad (1)$$

The equation describing the time variation of the membrane potential U^j of the j th neuron writes

$$U_t^j = \gamma U_{t-1}^j + \sum_{i \in V_n} J_{ji}^n f(U_{t-1}^i) + \theta^j + B_{t-1}^j, \quad U_0^j = u_0^j, \quad j \in V_n \quad t = 1, \dots, T \quad (2)$$

$f : \mathbb{R} \rightarrow]0, 1[$ is a monotonically increasing Lipschitz continuous bijection. γ is in $[0, 1)$ and determines the time scale of the intrinsic dynamics of the neurons. The B_t^j s are i.i.d. Gaussian random variables distributed as $\mathcal{N}_1(0, \sigma^2)^{11}$. They represent the fluctuations of the neurons' membrane potentials. The θ^j s are i.i.d. as $\mathcal{N}_1(\bar{\theta}, \theta^2)$. They are independent of the B_t^i s and represent the current injected in the neurons. The u_0^j s are i.i.d. random variables each governed by law μ_I .

The J_{ij}^n s are the synaptic weights. J_{ij}^n represents the strength with which the 'presynaptic' neuron j influences the 'postsynaptic' neuron i . They arise from a stationary Gaussian random field specified by its mean \bar{J} and covariance function $\Lambda : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$, see [1,2].

We note J^n the $|V_n| \times |V_n|$ matrix of the synaptic weights, $J^n = (J_{ij}^n)_{i,j \in V_n}$.

where $\Psi : \mathcal{T} \rightarrow \mathcal{T}$ is the following affine bijection. Writing $v = \Psi(u)$, we define

$$\begin{cases} v_0 = \Psi_0(u) = u_0 \\ v_s = \Psi_s(u) = u_s - \gamma u_{s-1} - \bar{\theta} \quad s = 1, \dots, T. \end{cases} \quad (3)$$

For $v \in \mathcal{T}$, we write $\Psi^{-1}(v) = (\Psi^{-1}(v)_0, \dots, \Psi^{-1}(v)_T)$. The coordinate $\Psi^{-1}(v)_t$ is the affine function of v_s , $s = 0 \dots t$ obtained from equations (3)

$$\Psi^{-1}(v)_t = \sum_{i=0}^t \gamma^i v_{t-i} + \bar{\theta} \frac{\gamma^t - 1}{\gamma - 1}, \quad t = 0, \dots, T.$$

We extend Ψ to a mapping $\mathcal{T}^{\mathbb{Z}} \rightarrow \mathcal{T}^{\mathbb{Z}}$ componentwise and introduce the following notation.

1. We note $\mathcal{N}_p(m, \Sigma)$ the density of the p -dimensional Gaussian variable with mean m and covariance matrix Σ .

Définition 2.1 For each measure $\mu \in \mathcal{M}(\mathcal{T}^{V_n})$ or $\mathcal{M}_S(\mathcal{T}^{\mathbb{Z}})$ we define $\underline{\mu}$ to be $\mu \circ \Psi^{-1}$.

Note that the correspondence $\mu \rightarrow \underline{\mu}$ is an isomorphism.

We note $Q^{V_n}(J^n)$ the element of $\mathcal{M}(\mathcal{T}^{V_n})$ which is the law of the solution to (2) conditioned on J^n . We let Q^{V_n} be the law averaged with respect to the weights.

Finally we introduce the image law in terms of which the principal results of this paper are formulated.

Définition 2.2 Let $\Pi^n \in \mathcal{M}(\mathcal{M}_S(\mathcal{T}^{\mathbb{Z}}))$ be the image law of Q^{V_n} through the function $\hat{\mu}_n : \mathcal{T}^{V_n} \rightarrow \mathcal{M}_S(\mathcal{T}^{\mathbb{Z}})$ defined by (1):

$$\Pi^n = Q^{V_n} \circ \hat{\mu}_n^{-1}$$

3 Characterization of the unique minimum of the rate function

In [1], to each measure $\nu \in \mathcal{M}(\mathcal{T}^{\mathbb{Z}})$ we associate the measure, noted Q^ν of $\mathcal{M}(\mathcal{T}^{\mathbb{Z}})$ such that $\underline{Q}^\nu = \mu_I^{\mathbb{Z}} \otimes \underline{Q}_{1,T}^\nu$ where $\underline{Q}_{1,T}^\nu$, a Gaussian measure on $\mathcal{T}_{1,T}^{\mathbb{Z}}$ with spectral density $\sigma^2 \delta(\theta) + \tilde{K}^\nu(\theta)$. The spectral density \tilde{K}^ν is defined in [2]. We also define the rate function H^ν , which is a linear approximation of the functional Γ defined in [1] and satisfies the relation $H^\mu(\mu) = H(\mu)$. We prove the following lemma in [1].

Lemma 3.1 For $\mu, \nu \in \mathcal{M}_S(\mathcal{T}^{\mathbb{Z}})$, $H^\nu(\mu) = 0$ if and only if $\mu = Q^\nu$.

As stated in the following proposition, there exists a unique minimum μ_e of the rate function. We provide explicit equations for μ_e which would facilitate its numerical simulation.

Proposition 3.1 There is a unique distribution $\mu_e \in \mathcal{M}_S(\mathcal{T}^{\mathbb{Z}})$ which minimises H . This distribution satisfies $H(\mu_e) = 0$ which is equivalent to $\mu_e = Q^{\mu_e}$.

PROOF. The proof, which is found in [1], is an easy consequence of the explicit method we outline to actually calculate μ_e below in theorem 3.2. \square

We characterize the unique measure μ_e such that $\mu_e = Q^{\mu_e}$ in terms of its image $\underline{\mu_e}$. This characterization allows one to directly numerically calculate μ_e . Since $\underline{\mu_e}$ is Gaussian, the problem becomes that of defining the latest entries of \bar{K}^{μ_e} and c^{μ_e} in terms of previous ones. Hence we characterize $\underline{\mu_e}$ recursively (in time), by providing a method of determining $\underline{\mu_{e0,t}}$ in terms of

$\underline{\mu}_{e0,t-1}$. Let $K_{(t-1,s-1)}^{\mu_e,l}$ be the $(t-1) \times (s-1)$ submatrix of $K^{\mu_e,l}$ composed of the rows from times 1 to $t-1$ and the columns from times 1 to $s-1$. Let the measure $\underline{\mu}_{e0,t}^{V_0} \in \mathcal{M}(\mathcal{T}_{0,t})$ be given by

$$\underline{\mu}_{e0,t}^{V_0}(dv) = \mu_I(dv_0) \otimes \mathcal{N}_t(c_{1,t}^{\mu_e}, \sigma^2 \text{Id}_t + K_{(t,t)}^{\mu_e,0}) dv_{1,t},$$

and $\underline{\mu}_{e(0,t),(0,s)}^{(0,l)} \in \mathcal{M}(\mathcal{T}_{0,t} \times \mathcal{T}_{0,s})$ be given by

$$\begin{aligned} \underline{\mu}_{e(0,t),(0,s)}^{(0,l)}(dv_{0,t}^0 dv_{0,s}^l) &= \mu_I(dv_0^0) \otimes \mu_I(dv_0^l) \otimes \\ &\quad \mathcal{N}_{t+s}((c_{1,t}^{\mu_e}, c_{1,s}^{\mu_e}), \sigma^2 \text{Id}_{t+s} + K_{(t,s)}^{\mu_e,(0,l)}) dv_{1,t}^0 dv_{1,s}^l, \end{aligned}$$

where

$$K_{(t,s)}^{\mu_e,(0,l)} = \begin{bmatrix} K_{(t,t)}^{\mu_e,0} & K_{(t,s)}^{\mu_e,l} \\ \dagger K_{(t,s)}^{\mu_e,l} & K_{(s,s)}^{\mu_e,0} \end{bmatrix},$$

and the \dagger sign represents the transpose of a matrix or vector.

The inductive method for calculating $\underline{\mu}_e$ is outlined in the theorem below.

Theorem 3.2 *We may characterise $\underline{\mu}_e$ inductively as follows. Initially $\underline{\mu}_{e0} = \mu_I^{\mathbb{Z}}$. Given that we have a complete characterisation of $\{\underline{\mu}_{e(0,t-1),(0,t-1)}^{(0,l)}, \underline{\mu}_{e0,t-1}^{V_0} : l \in \mathbb{Z}\}$, we may characterise $\{\underline{\mu}_{e(0,t),(0,t)}^{(0,l)}, \underline{\mu}_{e0,t}^{V_0} : l \in \mathbb{Z}\}$ according to the following identities. For $s \in [1, t]$,*

$$c_s^{\mu_e} = \bar{J} \int_{\mathbb{R}^s} f(\Psi^{-1}(v)_{s-1}) \underline{\mu}_{e0,s-1}^{V_0}(dv). \quad (4)$$

For $1 \leq r, s \leq t$, $K_{rs}^{\mu_e,k} = \theta^2 \delta_k 1_T^\dagger 1_T + \sum_{l=-\infty}^{\infty} \Lambda(k, l) M_{rs}^{\mu_e,l}$. Here, for $p = \max(r-1, s-1)$,

$$M_{rs}^{\mu_e,0} = \int_{\mathbb{R}^{p+1}} f(\Psi^{-1}(v)_{r-1}) \times f(\Psi^{-1}(v)_{s-1}) \underline{\mu}_{e0,p}^{V_0}(dv), \quad (5)$$

and for $l \neq 0$

$$M_{rs}^{\mu_e,l} = \int_{\mathbb{R}^r \times \mathbb{R}^s} f(\Psi^{-1}(v^0)_{r-1}) \times f(\Psi^{-1}(v^l)_{s-1}) \underline{\mu}_{e(0,r-1),(0,s-1)}^{(0,l)}(dv^0 dv^l). \quad (6)$$

Remark 1 From a practical point of view the t -dimensional integral in (4) and the $\max(r, s)$ -dimensional integral in (5) can be reduced by a change of variable to at most two dimensions. Similarly the $r+s$ -dimensional integral in (6) can be reduced to at most four dimensions.

Remark 2 If we make the biologically realistic assumption that the synaptic weights are not correlated beyond a certain correlation distance $d \geq 0$, $\Lambda(k, l) = 0$ if k or l does not belong to V_d it is seen that the matrixes $K_{rs}^{\mu_e,k}$ are 0 as soon as $k \notin V_d$: thus in this case the asymptotic description of the network of neurons is sparse.

4 Convergence results

We use the Large Deviation Principle proved in [2,1] to establish convergence results for the measures Π^n , Q^{V_n} and $Q^{V_n}(J^n)$.

Theorem 4.1 Π^n converges weakly to δ_{μ_e} , i.e., for all $\Phi \in \mathcal{C}_b(\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}}))$,

$$\lim_{n \rightarrow \infty} \int_{\mathcal{T}^{V_n}} \Phi(\hat{\mu}_n(u)) Q^{V_n}(du) = \Phi(\mu_e).$$

Similarly,

$$\lim_{n \rightarrow \infty} \int_{\mathcal{T}^{V_n}} \Phi(\hat{\mu}_n(u)) Q^{V_n}(J^n)(du) = \Phi(\mu_e) \quad J \text{ almost surely}$$

PROOF. The proof of the first result follows directly from the existence of an LDP for the measure Π^n , see theorem 3.1 in [2], and is a straightforward adaptation of the one in [3, Theorem 2.5.1]. The proof of the second result uses the same method, making use of theorem 4.2 below. \square

We can in fact obtain the following quenched convergence analogue of the usual lower bound inequality in the definition of a Large Deviation Principle.

Theorem 4.2 For each closed set F of $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ and for almost all J

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{|V_n|} \log [Q^{V_n}(J^n)(\hat{\mu}_n \in F)] \leq - \inf_{\mu \in F} H(\mu).$$

PROOF. The proof is a combination of Tchebyshev's inequality and the Borel-Cantelli lemma and is an adaptation of the one in [3, Theorem 2.5.4, Corollary 2.5.6]. \square

We define $\check{Q}^{V_n}(J^n) = \frac{1}{|V_n|} \sum_{j \in V_n} Q^{V_n}(J^n) \circ \mathcal{S}^{-j}$, where we recall the shift operator \mathcal{S} . Clearly $\check{Q}^{V_n}(J^n)$ is in $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{V_n})$. We define \check{Q}^{V_n} to be the expectation of $\check{Q}^{V_n}(J^n)$, with respect to the synaptic weights J .

Corollary 4.3 Fix m and let $n > m$. For almost every J and all $h \in \mathcal{C}_b(\mathcal{T}^{V_m})$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathcal{T}^{V_m}} h(u) \check{Q}^{V_n, V_m}(J^n)(du) &= \int_{\mathcal{T}^{V_m}} h(u) \mu_e^{V_m}(du). \\ \lim_{n \rightarrow \infty} \int_{\mathcal{T}^{V_m}} h(u) Q^{V_n, V_m}(du) &= \int_{\mathcal{T}^{V_m}} h(u) \mu_e^{V_m}(du). \end{aligned}$$

That is, the V_m^{th} marginals $\check{Q}^{V_n, V_m}(J^n)$ and Q^{V_n, V_m} of respectively $\check{Q}^{V_n}(J^n)$ and \check{Q}^{V_n} converge weakly to $\mu_e^{V_m}$ as $n \rightarrow \infty$.

PROOF. It is sufficient to apply theorem 4.1 in the case where Φ in $\mathcal{C}_b(\mathcal{M}_S(\mathcal{T}^\mathbb{Z}))$ is defined by

$$\Phi(\mu) = \int_{\mathcal{T}^{V_m}} h d\mu^{V_m}$$

and to use the fact that $Q^{V_n}, \check{Q}^{V_n}(J) \in \mathcal{M}_S(\mathcal{T}^{V_n})$. \square

We now prove the following ergodic-type theorem. We may represent the ambient probability space by \mathfrak{W} , where $\omega \in \mathfrak{W}$ is such that $\omega = (J_{ij}, B_t^j, u_0^j)$, where $i, j \in \mathbb{Z}$ and $0 \leq t \leq T-1$, recall (2). We denote the probability measure governing ω by \mathfrak{P} . Let $u^{(n)}(\omega) \in \mathcal{T}^{V_n}$ be defined by (2). As an aside, we may then understand $Q^{V_n}(J^n)$ to be the conditional law of \mathfrak{P} on $u^{(n)}(\omega)$, for given J^n .

Theorem 4.4 *Fix $m > 0$ and let $h \in C_b(\mathcal{T}^{V_m})$. For $u^{(n)}(\omega) \in \mathcal{T}^{V_n}$ (where $n > m$) \mathfrak{P} almost surely,*

$$\lim_{n \rightarrow \infty} \frac{1}{|V_n|} \sum_{j \in V_n} h\left(\pi^{V_m}(\mathcal{S}^j u^{(n)}(\omega))\right) = \int_{\mathcal{T}^{V_m}} h(u) d\mu_e^{V_m}(u), \quad (7)$$

where π^{V_m} is the projection onto V_m . Hence $\hat{\mu}_n(u^{(n)}(\omega))$ converges \mathfrak{P} -almost-surely to μ_e .

PROOF. Our proof is an adaptation of [3]. We may suppose without loss of generality that $\int_{\mathcal{T}^{V_m}} h(u) d\mu_e^{V_m}(u) = 0$. For $p > 1$ let

$$F_p = \left\{ \mu \in \mathcal{M}_S(\mathcal{T}^\mathbb{Z}) \mid \left| \int_{\mathcal{T}^{V_m}} h(u) \mu^{V_m}(du) \right| \geq \frac{1}{p} \right\}.$$

Since $\mu_e \notin F_p$, but it is the unique zero of H , it follows that $\inf_{F_p} H = m > 0$. Thus by theorem 3.1 in [2] there exists an n_0 , such that for all $n > n_0$,

$$Q^{V_n}(\hat{\mu}_n \in F_p) \leq \exp(-m|V_n|).$$

However

$$\mathfrak{P}(\omega | \hat{\mu}_n(u^{(n)}(\omega)) \in F_p) = Q^{V_n}(u | \hat{\mu}_n(u) \in F_p).$$

Thus

$$\sum_{n=0}^{\infty} \mathfrak{P}(\omega | \hat{\mu}_n(u^{(n)}(\omega)) \in F_p) < \infty.$$

We may thus conclude from the Borel-Cantelli Lemma that \mathfrak{P} almost surely, for every $\omega \in \mathfrak{W}$, there exists n_p such that for all $n \geq n_p$,

$$\left| \frac{1}{|V_n|} \sum_{j \in V_n} h\left(\pi^{V_m}(\mathcal{S}^j u^{(n)}(\omega))\right) \right| \leq \frac{1}{p}.$$

This yields (7) because p is arbitrary. The convergence of $\hat{\mu}_n(u^{(n)}(\omega))$ is a direct consequence of (7), since this means that each of the V_m^{th} marginals converge. \square

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